Local twistors and the conformal field equations

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July 22, 1999

Abstract

This letter establishes the connection between Friedrich's conformal field equations and the conformally invariant formalism of local twistors.

The conformal field equations as derived by Friedrich [7] have proved to be a valuable tool in both analytical and numerical work in general relativity. Not only has it been possible to derive global existence theorems for solutions of Einstein's field equations [8, 9], thereby obtaining rigorous results about the global properties of asymptotically flat space-times. It was also successfully demonstrated that the conformal field equations provide a well-defined and very well-behaved system of equations for numerical purposes ([11, 12, 3, 4, 2, 5]).

The success of this formulation of the Einstein equations is mainly due to the fact that the conformal field equations are *conformally invariant*. This allows for the inclusion of the parts of a space-time $\widetilde{M}$ which are "at infinity" with respect to the "physical metric" $\tilde{g}$. The extension into the "unphysical" space-time manifold M with metric g is achieved by embedding $\widetilde{M}$ into M in such way that on the image in M (which we identify with $\widetilde{M}$) the relation $g = \Omega^2 \tilde{g}$ holds, where Ω is a conformal factor, i.e., a non-negative scalar function on M with the property that it is strictly positive on $\widetilde{M}$. On the set $\mathcal{J} := \{\Omega = 0\}$ one imposes the condition $d\Omega \neq 0$, by which $\mathcal{J}$ becomes a regular three-dimensional hypersurface of M. For more information on this construction we refer to [13] (see also [1] for a recent review).

The physical metric $\tilde{g}$ on $\widetilde{M}$ is therefore replaced by a pair (g, Ω) on M, which gives rise to $\tilde{g} = \Omega^{-2} g$ at points of M with $\Omega \neq 0$, but makes sense also on points where Ω vanishes. Obviously, this relationship is not one-to-one, as there are many pairs (g, Ω) which give rise to the same metric $\tilde{g}$. In fact, we need to regard (g, Ω) as being equivalent to $(\theta^2 g, \theta\Omega)$ for any strictly positive function

θ on M. Thus, every equation for $\tilde{g}$ has the property that, when expressed in terms of a pair (g, Ω), it is invariant under the rescaling $g \mapsto \theta^2 g$, $\Omega \mapsto \theta\Omega$. This is the conformal invariance of the equation in question.

The purpose of this note is to show how the conformal field equations of Friedrich can be expressed in the manifestly conformally invariant formalism of local twistors [14].

We start by writing down the conformal field equations. Using the conventions of [14] throughout, we define $P_{ab} := -\frac{1}{2}\left(R_{ab} - 4\Lambda g_{ab}\right)$, where R_{ab} is the Ricci tensor of the metric g_{ab} with scalar curvature $R = 24\Lambda$. Furthermore, we define $d^a{}_{bcd} := \Omega^{-1} C^a{}_{bcd}$ and $s := -\frac{1}{4}\left(\Box\Omega - 4\Lambda\Omega\right)$. Note ([13]), that as a consequence of the smoothness of $\mathcal{J}$, the Weyl tensor $C^a{}_{bcd}$ vanishes on $\mathcal{J}$ so that the "gravitational field" $d^a{}_{bcd}$ is regular there. With these variables the conformal field equations can be expressed on M as follows:

$$\nabla_a P_{bc} - \nabla_b P_{ac} = -\nabla_e \Omega d^e{}_{cab}, \tag{1}$$

$$\nabla_a d^a{}_{bcd} = 0, \tag{2}$$

$$\nabla_a \nabla_b \Omega + \Omega P_{ab} + s g_{ab} = 0, \tag{3}$$

$$\nabla_a s - P_{ab} \nabla^a \Omega = 0, \tag{4}$$

$$2\Omega s - \nabla_a \Omega \nabla^a \Omega = \lambda/3. \tag{5}$$

The covariant derivative operator ∇ is the Levi-Civita connection of the metric g. Several remarks are in order:

1. These equations are easily checked to be conformally invariant.

2. If one takes $g = \tilde{g}$ and $\Omega = 1$ then the equations turn out to be equivalent on $\widetilde{M}$ to the Einstein vacuum equations with cosmological constant λ together with the Bianchi identity for the Weyl tensor.

3. When these equations are supplemented by the first and second Cartan structure equations then one can derive a first order system of equations for a tetrad, the connection, the curvature and the conformal factor.

4. Upon introduction of suitable "gauge source functions" this system can be decomposed into a symmetric hyperbolic system of evolution equations and a set of constraint equations. The constraints are propagated by the evolution. This is the basis for the analytical and numerical applications mentioned above.

We now want to briefly discuss the concept of a local twistor. Let (M, g) be any (four-dimensional) Lorentzian space-time. Let $\mathbb{T}^\alpha(M)$ denote a fibre bundle over M with each fibre isomorphic to twistor space $\mathbb{T}^\alpha$. In the usual manner, we may construct the Grassmann bundle $G_2(\mathbb{T}^\alpha)(M)$ of two-dimensional subspaces of $\mathbb{T}^\alpha$ over M. Then the structure of $\mathbb{T}^\alpha(M)$ is fixed by the requirement that the fibre of $G_2(\mathbb{T}^\alpha)(M)$ over any point be isomorphic to the (complexified, compactified) Minkowski vector space T_pM. This is nothing but the usual Klein correspondence which allows the identification of points of Minkowski space with two-dimensional subspaces of twistor space. Each element of the fibre of $\mathbb{T}^\alpha(M)$ over p is called a local twistor at p and a section Z^α of $\mathbb{T}^\alpha(M)$ is called a local twistor (field).

We may identify each tangent vector at $p \in M$ with a two-dimensional subspace of the fibre of $\mathbb{T}^\alpha(M)$ over p, which in turn can be identified with a bitwistor $V^{\alpha\beta}$ up to scale. Real tangent vectors correspond to bitwistors which satisfy the conditions

$$V^{\alpha\beta} = -V^{\beta\alpha}, \qquad V_{\alpha\beta} = \overline{V}_{\alpha\beta}.$$

Here, the bar denotes complex conjugation which takes twistors to dual twistors and $V_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}V^{\gamma\delta}$, where $\varepsilon_{\alpha\beta\gamma\delta}$ is the four-dimensional volume on twistor space.

Each fibre of $\mathbb{T}^\alpha(M)$ can be considered as a direct sum of two two-dimensional spin spaces. However, this decomposition depends on the conformal scale: if, for a given metric g a local twistor is represented by $(\omega^A, \pi_{A'})$, then for the conformally related metric $\theta^2 g$ this same twistor is represented by $(\omega^A, \pi_{A'} + i\Upsilon_{AA'}\omega^A)$, where $\Upsilon_a = \theta^{-1}\nabla_a\theta$. We write $Z^\alpha = (\omega^A, \pi_{A'})$, when the metric g is understood.

There exists a natural connection D on $\mathbb{T}^\alpha(M)$, the local twistor transport. It is defined in terms of the representing spinor fields by

$$DZ^\alpha = \left(d\omega^A + i\theta^{AA'}\pi_{A'}, d\pi_{A'} + iP_{ABA'B'}\theta^{BB'}\omega^A\right).$$

Here, the one-form $\theta^{AA'}$ is the van der Waerden one-form or soldering form while $P_{ABA'B'}$ is the spinor form of P_{ab} defined above. It is easily checked that this has the right conformal transformation properties. The curvature of D can be obtained as usual from $D^2 Z^\alpha = -iK_\beta{}^\alpha Z^\beta$.

The fact that the tangent spaces of M are not affine spaces but vector spaces with a preferred origin implies the existence of a global section $X^{\alpha\beta}$ (unique up to scale), representing the zero-section of $T(M)$. Equivalently, one can think of $X^{\alpha\beta}$ as representing the "current point" p in the fibre over p. This "origin twistor" is simple, i.e., it satisfies $X^{\alpha[\beta}X^{\gamma\delta]} = X^{[\alpha\beta}X^{\gamma\delta]} = 0$. We have

$$X^{\alpha\beta} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \varepsilon_{A'B'} \end{array}\right).$$

Its covariant derivative is easily computed as

$$DX^{\alpha\beta} = \left(\begin{array}{c|c} 0 & i\theta^A{}_{B'} \\ \hline -i\theta_{A'}{}^B & 0 \end{array}\right). \tag{6}$$

Thus, $DX^{\alpha\beta}$ assumes a rôle similar to the soldering form. As a consequence of (6) we have the condition

$$D^2 X^{\alpha\beta} = 0, \tag{7}$$

which can be interpreted as stating a torsion free condition for the local twistor connection. From (7) we conclude that $K_\alpha{}^\beta$ has the form

$$K_\alpha{}^\beta = \left(\begin{array}{c|c} i\Psi_A{}^B & \Phi_{AB'} \\ \hline 0 & -i\overline{\Psi}^{A'}{}_{B'} \end{array}\right)$$

Here $\Psi_{AB} = \Psi_{ABCD}\Sigma^{CD}$ contains only the Weyl curvature spinor while $\Phi_{AA'} = \nabla_{C(C'}P^C{}_{D')AA'}\Sigma^{C'D'} + c.c.$ contains only the Ricci curvature. The anti-self-dual two-form Σ^{AB} is defined by $\Sigma^{AB} = \theta^A{}_{A'} \wedge \theta^{BA'}$.

The Einstein equations themselves are not conformally invariant. Therefore, one needs to introduce an additional structural element in order to reduce the structure group from the conformal group to the Poincaré group. This is done conventionally by postulating the existence of an infinity twistor. This is a real bitwistor (field) $I^{\alpha\beta}$ which is to represent the "point at infinity". Finite points represented by $U^{\alpha\beta}$ have the property that $U_{\alpha\beta}I^{\alpha\beta} \neq 0$ while for points at infinity this quantity vanishes.

In terms of its representing spinor fields the infinity twistor has the form

$$I^{\alpha\beta} = \left(\begin{array}{c|c} f\varepsilon^{AB} & i\Gamma^{A}{}_{B'} \\ \hline -i\Gamma_{A'}{}^{B} & g\varepsilon_{A'B'} \end{array}\right)$$

with real functions f, g, and a hermitian spinor field $\Gamma_{AA'}$. Its covariant derivative has the form

$$DI^{\alpha\beta} = \left(\begin{array}{c|c} E\varepsilon^{AB} & iE^{A}{}_{B'} \\ \hline -iE_{A'}{}^{B} & E'\varepsilon_{A'B'} \end{array}\right)$$

for two one-forms E and E' and a hermitian one-form $E_{AA'}$. These forms are given by

$$E = df - \Gamma_{AA'}\theta^{AA'}, \tag{8}$$

$$E_{AA'} = d\Gamma_{AA'} + g\theta_{AA'} + P_{ABA'B'}\theta^{BB'}, \tag{9}$$

$$E' = dg - P_{ABA'B'}\theta^{BB'}\Gamma^{AA'}. \tag{10}$$

Since $X^{\alpha\beta}$ represents the current point and since $X_{\alpha\beta}I^{\alpha\beta} = 2f$ we require that f vanishes at infinity, i.e., at those points where $\Omega = 0$. Thus, we make the ansatz $f = \Omega$. Comparison of equations (8–10) with the conformal field equations (3) and (4) then shows, that these equations are exactly equivalent to the equation

$$DI^{\alpha\beta} = 0, \tag{11}$$

provided we make the identifications $f \leftarrow \Omega$, $\Gamma_{AA'} \leftarrow \nabla_{AA'}\Omega$ and $g \leftarrow s$. As a consequence of equation (11) we have $D\left(I^{\alpha\beta}I_{\alpha\beta}\right) = 0$ and hence the equation

$$I^{\alpha\beta}I_{\alpha\beta} = 2\Omega s - \nabla_{AA'}\Omega\nabla^{AA'}\Omega = \text{const.} = \lambda/3$$

i.e., equation (5). We note, that the infinity twistor has the property

$$I^{\alpha\gamma}I_{\beta\gamma} = \frac{\lambda}{12}\delta^{\alpha}_{\beta}. \tag{12}$$

Before we consider the curvature equations (1) and (2) we need to discuss some more properties of the origin and infinity twistors. We have the

Lemma 1 *The two twistors $U_\alpha{}^\beta = X_{\alpha\gamma}I^{\beta\gamma}$ and $P_\alpha{}^\beta = I_{\alpha\gamma}X^{\beta\gamma}$ possess the following properties:*

(i) $P_\alpha{}^\gamma P_\gamma{}^\beta = \Omega P_\alpha{}^\beta$, $\qquad U_\alpha{}^\gamma U_\gamma{}^\beta = \Omega U_\alpha{}^\beta$,

(ii) $P_\alpha{}^\gamma U_\gamma{}^\beta = U_\alpha{}^\gamma P_\gamma{}^\beta = 0$,

(iii) At points with $\Omega \neq 0$ both $\Omega^{-1}P_\alpha{}^\beta$ and $\Omega^{-1}Q_\alpha{}^\beta$ are projectors onto two-dimensional subspaces of $\mathbb{T}^\alpha$.

(iv) $P_\alpha{}^\beta + U_\alpha{}^\beta = \Omega\delta_\alpha{}^\beta$.

The proof of (i) is immediate when one uses the fact that $X^{\alpha\beta}$ is simple, which implies $X_{\alpha[\beta}X_{\gamma]\delta} = -\frac{1}{2}X_{\alpha\delta}X_{\beta\gamma}$. Property (ii) follows from direct calculation. The projector property (iii) follows from (i) and the fact that $P_\alpha{}^\alpha = Q_\alpha{}^\alpha = 2\Omega$. Finally, (iv) follows from (ii) and the four-dimensionality of twistor space.

Next, we consider the curvature equations (1) and (2). The definition of the gravitational field tensor $d^a{}_{bcd}$ suggests that we try to write $K_\alpha{}^\beta = D^\beta{}_{\alpha\gamma\delta}I^{\gamma\delta}$ for some $D^\beta{}_{\alpha\gamma\delta}$ which has to be determined. To begin with we have the following

Lemma 2 *Consider the space $\mathbb{V}_\alpha{}^\beta$ of twistors $E_\alpha{}^\beta$, satisfying the equations*

$$E_\gamma{}^{[\beta}X^{\alpha]\gamma} = 0, \qquad E_\gamma{}^{[\beta}I^{\alpha]\gamma} = 0.$$

Then the map

$$\omega : E_\alpha{}^\beta \mapsto I_{\alpha\gamma}E_\delta{}^\gamma X^{\delta\beta} + X_{\alpha\gamma}E_\delta{}^\gamma I^{\delta\beta}$$

maps $\mathbb{V}_\alpha{}^\beta$ into itself. Furthermore, it is an isomorphism at points with $\Omega \neq 0$.

The proof is straightforward once we have made the following observation. For any $E_\alpha{}^\beta \in \mathbb{V}_\alpha{}^\beta$ we have

$$I_{\alpha\gamma}E_\delta{}^\gamma X^{\delta\beta} = I_{\alpha\gamma}E_\delta{}^\beta X^{\delta\gamma} = P_\alpha{}^\delta E_\delta{}^\beta$$

and similarly for the other term, which equals $U_\alpha{}^\delta E_\delta{}^\beta$. Thus, ω maps $E_\alpha{}^\beta \mapsto \Omega E_\alpha{}^\beta$ which is obviously an isomorphism if $\Omega \neq 0$.

Finally, we find that ω does exactly we set out to show. For we have the

Proposition 1 *The curvature $K_\alpha{}^\beta$ of the local twistor connection can be written as*

$$K_\alpha{}^\beta = I_{\alpha\gamma}E_\delta{}^\gamma X^{\delta\beta} + X_{\alpha\gamma}E_\delta{}^\gamma I^{\delta\beta}$$

for some uniquely determined hermitian, tracefree twistor $E_\delta{}^\gamma$. This twistor satisfies the equation

$$I_{\alpha\gamma}DE_\delta{}^\gamma X^{\delta\beta} + X_{\alpha\gamma}DE_\delta{}^\gamma I^{\delta\beta} + I_{\alpha\gamma}E_\delta{}^\gamma DX^{\delta\beta} + DX_{\alpha\gamma}E_\delta{}^\gamma I^{\delta\beta} = 0,$$

or, equivalently,

$$DE_\alpha{}^\beta + \frac{D\Omega}{\Omega}E_\alpha{}^\beta = 0.$$

A further consequence of (11) is

$$K_\gamma{}^{[\alpha}I^{\beta]\gamma} = 0.$$

Thus, $K_\alpha{}^\beta$ is a two-form with values in $\mathbb{V}_\alpha{}^\beta$ and by lemma 2 it is of the form given above at points with $\Omega \neq 0$. The fact that $E_\delta{}^\gamma$ is hermitian and tracefree follows from hermiticity and vanishing trace of $K_\alpha{}^\beta$. We extend this form for the curvature by continuity to the points with $\Omega = 0$. Then we obtain, that for regular $E_\delta{}^\gamma$ some parts of the local twistor curvature must vanish. This is in complete analogy to the conformal field equations where the regularity of $d^a{}_{bcd}$ implies that the Weyl curvature vanishes on $\mathcal{J}$. The first form of the field equation for $E_\alpha{}^\beta$ follows from the Bianchi identity $DK_\alpha{}^\beta = 0$, while the second form arises either by simple manipulation of the first, using the fact that the infinity twistor is covariantly constant, or by observing that $K_\alpha{}^\beta = \Omega E_\alpha{}^\beta$.

In order to make contact with the conformal field equations we now determine the space $\mathbb{V}_\alpha{}^\beta$ of hermitian, tracefree twistors satisfying the equations $E_\gamma{}^{[\beta}X^{\alpha]\gamma} = 0$ and $E_\gamma{}^{[\beta}I^{\alpha]\gamma} = 0$. It is easy to check that a hermitian $E_\alpha{}^\beta$ with $E_\gamma{}^{[\beta}X^{\alpha]\gamma} = 0$ is of the form

$$E_\alpha{}^\beta = \left(\begin{array}{c|c} E_A{}^B & F_{AB'} \\ \hline 0 & \bar{E}^{A'}{}_{B'} \end{array}\right)$$

with a symmetric spinor E_{AB} and a hermitian spinor $F_{AA'}$. It is obviously tracefree. The final equation $E_\gamma{}^{[\beta}I^{\alpha]\gamma} = 0$ imposes the additional condition

$$\Omega F_{AA'} = i\Gamma_A{}^{C'}\bar{E}_{C'A'} - i\Gamma_{A'}{}^C E_{CA},$$

which, assuming regularity and redefining E_{AB}, can be satisfied by writing

$$E_\alpha{}^\beta = \left(\begin{array}{c|c} i\Omega E_A{}^B & \Gamma_{B'}{}^C E_{CA} - \Gamma_A{}^{C'}\bar{E}_{C'B'} \\ \hline 0 & -i\Omega\bar{E}^{A'}{}_{B'} \end{array}\right).$$

This is the form that any twistor in $\mathbb{V}_\alpha{}^\beta$ obeys. Since the curvature is a two-form with values in this space, we can write it also in this way, where now E_{AB} and $F_{AA'}$ are spinor valued two-forms. Stripping off the basis two-forms we obtain the equations

$$\Psi_{ABCD} = \Omega E_{ABCD}, \tag{13}$$

$$\nabla_{C'(C}P^{C'}{}_{D)AA'} = \Gamma_{A'}{}^B E_{BACD}, \tag{14}$$

which are easily verified to be the spinorial equivalents of the definition of $d^a{}_{bcd}$ and equation (1).

The remaining equation is obtained from the field equation for $E_\alpha{}^\beta$. After a lengthy calculation one finds that the only remaining equation which is not identically satisfied due to earlier equations is

$$dE_B{}^A - \frac{1}{\Omega}\left(\Gamma_B{}^{B'}\bar{E}_{B'C'}\theta^{AC'} - \Gamma_{C'}{}^C E_{CB}\theta^{AC'} - \theta^{CC'}\Gamma_{CC'}E_B{}^A\right) = 0.$$

Here the first term in the parenthesis vanishes because of the symmetry of $\bar{E}_{A'B'C'D'}$, while the remaining two terms are seen to cancel each other after some manipulation. Thus, the field equation for $E_\alpha{}^\beta$ reduces to the single equation $dE_B{}^A = 0$, which when written in terms of components is

$$\nabla_{A'}{}^A E_{ABCD} = 0,$$

the spinor equivalent of the conformal field equation (2).

In summary, we have shown in this article the following

Theorem 1 *The contents of the conformal field equations (1–5) is equivalent to the existence of a covariantly constant "infinity twistor" $I^{\alpha\beta}$. Its algebraic properties determine the cosmological constant.*

This concludes our discussion of the conformal field equations in the version of [7]. Our future aim is to also find the relationship between the newer version of the conformal field equations and local twistors. In [10] it is shown that one can gain additional freedom by introducing an arbitrary Weyl connection, which respects the given conformal structure and expressing the equations in terms of this connection. This removes the field equation for the conformal factor Ω, which can therefore be fixed by other methods. The natural setting for these constructions is the normal conformal Cartan connection which, as was shown by Friedrich [6] coincides with the local twistor connection. Thus, we expect that there exists a very natural interpretation in terms of local twistors also for the more general form of the conformal field equations.

The authors wish to express their gratitude to the Erwin Schrödinger Institute in Vienna and to the Collegium Budapest where part of this work was made. JF thanks the Deutsche Forschungsgemeinschaft for financial support.

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